

MULTIPLICITY OF SOLUTIONS FOR FRACTIONAL SCHRÖDINGER SYSTEMS IN \mathbb{R}^N

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ABSTRACT. In this paper we deal with the following nonlocal systems of fractional Schrödinger equations

$$\begin{cases} \varepsilon^{2s}(-\Delta)^s u + V(x)u = Q_u(u, v) + \gamma H_u(u, v) & \text{in } \mathbb{R}^N \\ \varepsilon^{2s}(-\Delta)^s u + W(x)v = Q_v(u, v) + \gamma H_v(u, v) & \text{in } \mathbb{R}^N \\ u, v > 0 & \text{in } \mathbb{R}^N \end{cases}$$

where $\varepsilon > 0$, $s \in (0, 1)$, $N > 2s$, $(-\Delta)^s$ is the fractional Laplacian, $V : \mathbb{R}^N \rightarrow \mathbb{R}$ and $W : \mathbb{R}^N \rightarrow \mathbb{R}$ are continuous potentials, Q is an homogeneous C^2 -function with subcritical growth, $\gamma \in \{0, 1\}$ and $H(u, v) = \frac{1}{\alpha+\beta}|u|^\alpha|v|^\beta$ where $\alpha, \beta \geq 1$ are such that $\alpha + \beta = 2_s^*$.

We investigate the subcritical case ($\gamma = 0$) and the critical case ($\gamma = 1$), and by using Ljusternik-Schnirelmann theory, we relate the number of solutions with the topology of the set where the potentials V and W attain their minimum values.

1. INTRODUCTION

In the last decade a tremendous popularity has received the study of nonlinear partial differential equations involving fractional and nonlocal operators, due to the fact that such operators have great applications in many areas of the research such as crystal dislocation, finance, phase transitions, material sciences, chemical reactions, minimal surfaces; see for instance [22, 38] for more details. Motivated by the interest shared by the mathematical community in this topic, the aim of this paper is to investigate the existence and the multiplicity of positive solutions for the following nonlinear fractional Schrödinger system

$$\begin{cases} \varepsilon^{2s}(-\Delta)^s u + V(x)u = Q_u(u, v) + \gamma H_u(u, v) & \text{in } \mathbb{R}^N \\ \varepsilon^{2s}(-\Delta)^s u + W(x)v = Q_v(u, v) + \gamma H_v(u, v) & \text{in } \mathbb{R}^N \\ u, v > 0 & \text{in } \mathbb{R}^N, \end{cases} \quad (1.1)$$

where $\varepsilon > 0$ is a parameter, $s \in (0, 1)$, $N > 2s$, $V : \mathbb{R}^N \rightarrow \mathbb{R}$ and $W : \mathbb{R}^N \rightarrow \mathbb{R}$ are continuous potentials, Q is an homogeneous C^2 -function with subcritical growth, $\gamma \in \{0, 1\}$, and $H(u, v) = \frac{1}{\alpha+\beta}|u|^\alpha|v|^\beta$ where $\alpha, \beta \geq 1$ are such that $\alpha + \beta = 2_s^* = \frac{2N}{N-2s}$.

The nonlocal operator $(-\Delta)^s$ is the so-called fractional Laplacian operator which can be defined for any $u : \mathbb{R}^N \rightarrow \mathbb{R}$ smooth enough, by setting

$$(-\Delta)^s u(x) = -\frac{C(N, s)}{2} \int_{\mathbb{R}^N} \frac{u(x+y) - u(x-y) - 2u(x)}{|y|^{N+2s}} dy \quad (x \in \mathbb{R}^N)$$

where $C(N, s)$ is a dimensional constant depending only on N and s ; see for instance [22].

In the scalar case, the problem (1.1) becomes the well-known fractional Schrödinger equation

$$\varepsilon^{2s}(-\Delta)^s u + V(x)u = f(x, u) \text{ in } \mathbb{R}^N. \quad (1.2)$$

Key words and phrases. Fractional Schrödinger systems; variational methods; Ljusternik-Schnirelmann theory; positive solutions.

We recall that one of the main reasons of the study of (1.2), is related to the seek of standing wave solutions $\Phi(t, x) = u(x)e^{-\frac{ict}{\hbar}}$ for the following time-dependent fractional Schrödinger equation

$$i\hbar \frac{\partial \Phi}{\partial t} = \frac{\hbar^2}{2m} (-\Delta)^s \Phi + V(x)\Phi - g(|\Phi|)\Phi \text{ for } (t, x) \in \mathbb{R} \times \mathbb{R}^N. \quad (1.3)$$

The equation (1.3) has been proposed by Laskin [34, 35], and it is a fundamental equation of fractional quantum mechanics in the study of particles on stochastic fields modeled by Lévy processes. When $s = 1$, the equation (1.2) is reduced to the classical Schrödinger equation

$$-\varepsilon^2 \Delta u + V(x)u = f(u) \text{ in } \mathbb{R}^N, \quad (1.4)$$

which has been extensively investigated in the last twenty years by many authors; see for instance [2, 5, 10, 17, 21, 30, 40] and references therein.

Recently, the study of fractional Schrödinger equations has attracted the attention of many mathematicians. Felmer et al. [26] investigated existence, regularity and qualitative properties of positive solution to (1.2) when V is constant, and f is a smooth function with subcritical growth satisfying the Ambrosetti-Rabinowitz condition. Secchi [41, 42] proved some existence results for some nonlinear fractional Schrödinger equations under the assumptions that the nonlinearity is either of perturbative type or satisfies the Ambrosetti-Rabinowitz condition. Frank et al. [31] studied uniqueness and nondegeneracy of ground state solutions to (1.2) with $f(u) = |u|^\alpha u$, for all H^s -admissible powers $\alpha \in (0, \alpha^*)$. Chang and Wang [14] showed the existence of nontrivial solutions to (1.2) with $V(x) = 1$, and f is autonomous and verifies Berestycki-Lions type assumptions. Dávila et al. [19] obtained the existence of a multi-peak solution for a fractional Schrödinger equation with a bounded smooth positive potential, by using Lyapunov-Schmidt reduction method. Shang et al. [45] used variational methods to deal with the multiplicity of solutions of a fractional Schrödinger equation with critical growth, and with a continuous and positive potential V . Pucci et al. [39] established via Mountain Pass Theorem and Ekeland variational principle, the existence of multiple solutions for a Kirchhoff fractional Schrödinger equation driven by the fractional p -Laplacian, a nonlinearity $f(x, u)$ satisfying the Ambrosetti-Rabinowitz condition, a positive potential $V(x)$ verifying suitable assumptions, and in presence of a perturbation term. Figueiredo and Siciliano [29] obtained a multiplicity result by means of the Lyusternik-Shnirelman and Morse theories for (1.2) involving a superlinear nonlinearity with subcritical growth. Alves and Miyagaki in [3] (see also [8]) dealt with the existence and the concentration of positive solutions to (1.2), via penalization method. We also mention the papers [6, 7, 18, 23, 24, 33, 37, 44, 47, 48] where the existence and the multiplicity of solutions to (1.2) have been investigated under various assumptions on the potential V and the nonlinearity f , by using suitable variational and topological approaches.

Particularly motivated by the papers [29, 44], in this work we aim to extend the multiplicity results for both subcritical and critical cases obtained for the scalar equation (1.2), to the case of the systems. More precisely, we generalize in nonlocal setting, some existence and multiplicity results appeared in [4, 9, 27, 28], in which the authors studied elliptic systems of the type

$$\begin{cases} -\varepsilon^2 \Delta u + V(x)u = Q_u(u, v) + \gamma H_u(u, v) & \text{in } \mathbb{R}^N \\ -\varepsilon^2 \Delta v + W(x)v = Q_v(u, v) + \gamma H_v(u, v) & \text{in } \mathbb{R}^N \\ u, v > 0 & \text{in } \mathbb{R}^N. \end{cases}$$

To the best of our knowledge, there are few results on the nonlocal systems involving the fractional Laplacian in the literature [15, 16, 25, 32, 36, 49], and the results presented here seems to be new in nonlocal framework.

In order to state the main theorems obtained in this work, we come back to our problem (1.1), and we introduce the assumptions on the potentials V , W and the function Q .

Firstly, we define the following constants

$$V_0 = \inf_{x \in \mathbb{R}^N} V(x) \text{ and } W_0 = \inf_{x \in \mathbb{R}^N} W(x)$$

and

$$V_\infty = \liminf_{|x| \rightarrow \infty} V(x) \text{ and } W_\infty = \liminf_{|x| \rightarrow \infty} W(x).$$

Along the paper, we will assume the following conditions on V and W :

- (H1) $V_0 = W_0$, and $M = \{x \in \mathbb{R}^N : V(x) = W(x) = V_0\}$ is nonempty;
(H2) $V_0 < \max\{V_\infty, W_\infty\}$.

Regarding the function Q , we suppose that $Q \in C^2(\mathbb{R}_+^2, \mathbb{R})$ and verifying the following conditions:

- (Q1) there exists $q \in (2, 2_s^*)$ such that $Q(tu, tv) = t^q Q(u, v)$ for all $t > 0$, $(u, v) \in \mathbb{R}_+^2$;
(Q2) there exists $C > 0$ such that $|Q_u(u, v)| + |Q_v(u, v)| \leq C(u^{q-1} + v^{q-1})$ for all $(u, v) \in \mathbb{R}_+^2$;
(Q3) $Q_u(0, 1) = 0 = Q_v(1, 0)$;
(Q4) $Q_u(1, 0) = 0 = Q_v(0, 1)$;
(Q5) $Q_{uv}(u, v) > 0$ for all $(u, v) \in \mathbb{R}_+^2$.

Since we look for positive solutions of (1.1), we extend the function Q to the whole \mathbb{R}^2 by setting $Q(u, v) = 0$ if $u \leq 0$ or $v \leq 0$. We note that the q -homogeneity of Q , implies that the following identity holds:

$$qQ(u, v) = uQ_u(u, v) + vQ_v(u, v) \text{ for any } (u, v) \in \mathbb{R}^2. \quad (1.5)$$

Moreover, by using (Q2), we can see that there exists $C > 0$ such that

$$|Q(u, v)| \leq C(|u|^q + |v|^q) \text{ for any } (u, v) \in \mathbb{R}^2. \quad (1.6)$$

A typical example (see [20]) of function Q which satisfies the above assumptions is the following one. Let $p \geq 1$ and

$$\mathcal{P}_p(u, v) = \sum_{\alpha_i + \beta_i = p} a_i u^{\alpha_i} v^{\beta_i}$$

where $i \in \{1, \dots, k\}$, $\alpha_i, \beta_i \geq 1$ and $a_i \in \mathbb{R}$. The following functions and their possible combinations, with appropriate choice of the coefficients a_i , satisfy the assumptions (Q1)-(Q5) on Q

$$Q_1(u, v) = \mathcal{P}_q(u, v), \quad Q_2(u, v) = \sqrt[r]{\mathcal{P}_l(u, v)} \quad \text{and} \quad Q_3(u, v) = \frac{\mathcal{P}_{l_1}(u, v)}{\mathcal{P}_{l_2}(u, v)},$$

with $r = lq$ and $l_1 - l_2 = q$.

Now, we pass to state our main multiplicity results related to (1.1). When we take $\gamma = 0$ in (1.1), we have to deal with a system with subcritical growth, namely

$$\begin{cases} \varepsilon^{2s}(-\Delta)^s u + V(x)u = Q_u(u, v) & \text{in } \mathbb{R}^N \\ \varepsilon^{2s}(-\Delta)^s v + W(x)v = Q_v(u, v) & \text{in } \mathbb{R}^N \\ u, v > 0 & \text{in } \mathbb{R}^N. \end{cases} \quad (1.7)$$

Since we aim to relate the number of solutions of (1.7) with the topology of the set M of minima of the potential, it is worth to recall that if Y is a given closed set of a topological space X , we denote by $\text{cat}_X(Y)$ the Ljusternik-Schnirelmann category of Y in X , that is the least number of closed and contractible sets in X which cover Y .

With the above notations, the first main multiplicity result can be stated as follows.

Theorem 1.1. *Assume that (H1)-(H2) and (Q1)-(Q5) hold. Then, for any $\delta > 0$, there exists $\varepsilon_\delta > 0$ such that for any $\varepsilon \in (0, \varepsilon_\delta)$, the system (1.7) admits at least $\text{cat}_{M_\delta}(M)$ solutions.*

It is worth noting that, a common approach to deal with fractional nonlocal problems, is to make use of the Caffarelli-Silvestre method [13], which consists to transform via a Dirichlet-Neumann map, a given nonlocal problem into a local degenerate elliptic problem set in the half-space \mathbb{R}_+^{N+1} and with a nonlinear Neumann boundary condition. In this work, we prefer to analyze the problem directly in $H^s(\mathbb{R}^N)$, in order to adapt in our context some ideas developed in the case $s = 1$.

The proof of Theorem 1.1 is variational, and it is based on the method of the Nehari-manifold. After proving some compactness results for the functional associated to (1.7), and observing that the level

of compactness are deeply related to the behavior of the potentials V and W at infinity, we use the arguments developed in [11, 17], to compare the category of some sub-levels of the functional and the category of the set M . We recall that this type of approach is also used in the scalar case; see for instance [29, 44, 45].

In the second part of our paper, we consider the critical case $\gamma = 1$, that is

$$\begin{cases} \varepsilon^{2s}(-\Delta)^s u + V(x)u = Q_u(u, v) + \frac{\alpha}{\alpha+\beta}|u|^{\alpha-2}u|v|^\beta & \text{in } \mathbb{R}^N \\ \varepsilon^{2s}(-\Delta)^s u + W(x)v = Q_v(u, v) + \frac{\beta}{\alpha+\beta}|u|^\alpha|v|^{\beta-2}v & \text{in } \mathbb{R}^N \\ u, v > 0 & \text{in } \mathbb{R}^N \end{cases} \quad (1.8)$$

where $\alpha, \beta \geq 1$ are such that $\alpha + \beta = 2_s^*$.

In this context, we assume that Q verifies the following technical assumption:

(Q6) $Q(u, v) \geq \lambda u^{\tilde{\alpha}} v^{\tilde{\beta}}$ for any $(u, v) \in \mathbb{R}_+^2$ with $1 < \tilde{\alpha}, \tilde{\beta} < 2_s^*$, $\tilde{\alpha} + \tilde{\beta} = q_1 \in (2, 2_s^*)$, and λ verifying

- $\lambda > 0$ if either $N \geq 4s$, or $2s < N < 4s$ and $2_s^* - 2 < q_1 < 2_s^*$;
- λ is sufficiently large if $2s < N < 4s$ and $2 < q_1 \leq 2_s^* - 2$.

To obtain the multiplicity of positive solutions to (1.8), we proceed as in the subcritical case. Clearly, the lack of the compactness due to the presence of the critical Sobolev exponent, creates a further difficulty, and more accurate estimates are needed to localize the energy levels where the Palais-Smale condition fails. To circumvent this hitch, we combine the estimates obtained in [43] with some adaptations of the calculations in [1], which allow us to prove that the number

$$\tilde{S}_*(\alpha, \beta) = \inf_{u, v \in H^s(\mathbb{R}^N) \setminus \{(0,0)\}} \frac{\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 + |(-\Delta)^{\frac{s}{2}} v|^2 dx}{\left(\int_{\mathbb{R}^N} |u|^\alpha |v|^\beta dx \right)^{\frac{2}{2_s^*}}}$$

is strongly related to the best constant S_* of the Sobolev embedding $H^s(\mathbb{R}^N)$ into $L^{2_s^*}(\mathbb{R}^N)$, and plays a fundamental role when we have to study critical systems like (1.8).

Our second main result can be stated as follows.

Theorem 1.2. *Let us assume that (H1)-(H2) and (Q1)-(Q6) hold. If $\alpha, \beta \in [1, 2_s^*)$ are such that $\alpha + \beta = 2_s^*$, then for any $\delta > 0$, there exists $\varepsilon_\delta > 0$ such that for any $\varepsilon \in (0, \varepsilon_\delta)$, the system (1.8) possesses at least $\text{cat}_{M_\delta}(M)$ solutions.*

We conclude this introduction observing that our results complement the ones obtained in [29, 44], in the sense that now we are considering the multiplicity results in the case of systems.

The structure of the paper is the following. In Section 2 we give some preliminary facts about the fractional Sobolev spaces and we set up the variational framework. In Section 3 we deal with the autonomous problem related to (1.7). In Section 4 we prove some compactness results for the functional associated to (1.7). In Section 5 we present the proof of Theorem 1.1. In the last Section, we discuss the existence and the multiplicity of solutions for the system (1.1) in the critical case $\gamma = 1$.

2. PRELIMINARIES AND VARIATIONAL SETTING

In this section we collect some preliminary results about the fractional Sobolev spaces, and we introduce the functional setting.

For any $s \in (0, 1)$ we define $\mathcal{D}^{s,2}(\mathbb{R}^N)$ as the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to

$$\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 dx = \iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy,$$

that is

$$\mathcal{D}^{s,2}(\mathbb{R}^N) = \left\{ u \in L^{2_s^*}(\mathbb{R}^N) : \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 dx < \infty \right\}.$$

Let us introduce the fractional Sobolev space

$$H^s(\mathbb{R}^N) = \left\{ u \in L^2(\mathbb{R}^N) : \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 dx < \infty \right\}$$

endowed with the natural norm

$$\|u\|_{H^s(\mathbb{R}^N)} = \sqrt{\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 dx + \int_{\mathbb{R}^N} |u|^2 dx}.$$

For the convenience of the reader, we recall the following embeddings:

Theorem 2.1. [22] *Let $s \in (0, 1)$ and $N > 2s$. Then there exists a sharp constant $S_* = S(N, s) > 0$ such that for any $u \in H^s(\mathbb{R}^N)$*

$$\left(\int_{\mathbb{R}^N} |u|^{2^*_s} dx \right)^{\frac{2}{2^*_s}} \leq S_* \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 dx. \quad (2.1)$$

Moreover $H^s(\mathbb{R}^N)$ is continuously embedded in $L^q(\mathbb{R}^N)$ for any $q \in [2, 2^*_s]$ and compactly in $L^q_{loc}(\mathbb{R}^N)$ for any $q \in [2, 2^*_s)$.

We also recall the following Lions-compactness lemma.

Lemma 2.1. [26] *Let $N > 2s$. If (u_n) is a bounded sequence in $H^s(\mathbb{R}^N)$ and if*

$$\lim_{n \rightarrow \infty} \sup_{y \in \mathbb{R}^N} \int_{B_R(y)} |u_n|^2 dx = 0$$

where $R > 0$, then $u_n \rightarrow 0$ in $L^t(\mathbb{R}^N)$ for all $t \in (2, 2^*_s)$.

Now, we give the variational framework of problem (1.7). By using the change of variable $z \mapsto \varepsilon x$, we are led to consider the following problem

$$\begin{cases} (-\Delta)^s u + V(\varepsilon x)u = Q_u(u, v) & \text{in } \mathbb{R}^N \\ (-\Delta)^s u + W(\varepsilon x)v = Q_v(u, v) & \text{in } \mathbb{R}^N \\ u, v > 0 & \text{in } \mathbb{R}^N. \end{cases} \quad (2.2)$$

For any $\varepsilon > 0$, we introduce the fractional space

$$\mathbb{H}_\varepsilon = \{(u, v) \in H^s(\mathbb{R}^N) \times H^s(\mathbb{R}^N) : \int_{\mathbb{R}^N} (V(\varepsilon x)|u|^2 + W(\varepsilon x)|v|^2) dx < \infty\}.$$

endowed with the norm

$$\|(u, v)\|_\varepsilon^2 = \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 + |(-\Delta)^{\frac{s}{2}} v|^2 dx + \int_{\mathbb{R}^N} (V(\varepsilon x)u^2 + W(\varepsilon x)v^2) dx.$$

Let us introduce

$$\mathcal{J}_\varepsilon(u) = \frac{1}{2} \|(u, v)\|_\varepsilon^2 - \int_{\mathbb{R}^N} Q(u, v) dx$$

for any $(u, v) \in \mathbb{H}_\varepsilon$. We define the minimax level

$$c_\varepsilon = \inf_{(u, v) \in \mathcal{N}_\varepsilon} \mathcal{J}_\varepsilon(u, v)$$

where

$$\mathcal{N}_\varepsilon = \{(u, v) \in \mathbb{H}_\varepsilon \setminus \{(0, 0)\} : \langle \mathcal{J}'_\varepsilon(u, v), (u, v) \rangle = 0\}.$$

It is standard to check that \mathcal{J}_ε satisfies Mountain Pass geometry. Indeed, $\mathcal{J}_\varepsilon \in C^1(\mathbb{H}_\varepsilon, \mathbb{R})$ and $\mathcal{J}_\varepsilon(0, 0) = 0$. By using (1.6) and Theorem 2.1, we get for any $(u, v) \in \mathbb{H}_\varepsilon$

$$\mathcal{J}_\varepsilon(u, v) \geq \frac{1}{2} \|(u, v)\|_\varepsilon^2 - C \|(u, v)\|_\varepsilon^q,$$

so there exist $\mu, \rho > 0$ such that $\mathcal{J}_\varepsilon(u, v) \geq \rho$ for any $\|(u, v)\|_\varepsilon = \mu$. From (Q1), we can see that for any $(u, v) \in \mathbb{H}_\varepsilon \setminus \{(0, 0)\}$

$$\mathcal{J}_\varepsilon(tu, tv) = \frac{t^2}{2} \|(u, v)\|_\varepsilon^2 - t^q \int_{\mathbb{R}^N} Q(u, v) dx \rightarrow -\infty \text{ as } t \rightarrow \infty.$$

Finally, in view of (1.6), we can note that there exists $r > 0$ such that for any $\varepsilon > 0$

$$\|(u, v)\|_\varepsilon \geq r \text{ for any } (u, v) \in \mathcal{N}_\varepsilon. \quad (2.3)$$

Since \mathcal{J}_ε satisfies Mountain Pass geometry, we can use the homogeneity of Q to prove that c_ε can be alternatively characterized by

$$c_\varepsilon = \inf_{\gamma \in \Gamma_\varepsilon} \max_{t \in [0, 1]} \mathcal{J}_\varepsilon(\gamma(t)) = \inf_{(u, v) \in \mathbb{H}_\varepsilon \setminus \{(0, 0)\}} \max_{t \geq 0} \mathcal{J}_\varepsilon(tu, tv) > 0$$

where $\Gamma_\varepsilon = \{\gamma \in C([0, 1], \mathbb{H}_\varepsilon) : \gamma(0) = 0, \mathcal{J}_\varepsilon(\gamma(1)) < 0\}$. Moreover, for any $(u, v) \neq (0, 0)$, there exists a unique $t > 0$ such that $(tu, tv) \in \mathcal{N}_\varepsilon$. The maximum of the function $t \rightarrow \mathcal{J}_\varepsilon(tu, tv)$ for $t \geq 0$ is achieved at $t = \bar{t}$.

3. THE AUTONOMOUS PROBLEM WHEN $\gamma = 0$

In this section we establish an existence result for the autonomous problem associated to (1.7). Let us consider the following subcritical autonomous system

$$\begin{cases} (-\Delta)^s u + V_0 u = Q_u(u, v) & \text{in } \mathbb{R}^N \\ (-\Delta)^s u + W_0 v = Q_v(u, v) & \text{in } \mathbb{R}^N \\ u, v > 0 & \text{in } \mathbb{R}^N. \end{cases} \quad (3.1)$$

We set $\mathbb{H}_0 = H^s(\mathbb{R}^N) \times H^s(\mathbb{R}^N)$ endowed with the following norm

$$\|(u, v)\|_0^2 = \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 + |(-\Delta)^{\frac{s}{2}} v|^2 dx + \int_{\mathbb{R}^N} (V_0 u^2 + W_0 v^2) dx.$$

Let us introduce the functional $\mathcal{J}_0 : \mathbb{H}_0 \rightarrow \mathbb{R}$ defined as

$$\mathcal{J}_0(u, v) = \frac{1}{2} \|(u, v)\|_0^2 - \int_{\mathbb{R}^N} Q(u, v) dx.$$

Let

$$c_0 = \inf_{(u, v) \in \mathcal{N}_0} \mathcal{J}_0(u, v) = \inf_{(u, v) \in X_0 \setminus \{(0, 0)\}} \max_{t \geq 0} \mathcal{J}_0(tu, tv),$$

where

$$\mathcal{N}_0 = \{(u, v) \in \mathbb{H}_0 \setminus \{(0, 0)\} : \langle \mathcal{J}'_0(u, v), (u, v) \rangle = 0\}.$$

We begin proving the following useful lemma.

Lemma 3.1. *Let $\{(u_n, v_n)\} \subset \mathbb{H}_0$ be a bounded sequence such that $\mathcal{J}'_0(u_n, v_n) \rightarrow 0$. Then we have either*

- (i) $\|(u, v)\|_0 \rightarrow 0$, or
- (ii) *there exists a sequence $(y_n) \subset \mathbb{R}^N$ and $R, \gamma > 0$ such that*

$$\liminf_{n \rightarrow \infty} \int_{B_R(y_n)} (|u_n|^2 + |v_n|^2) dx \geq \gamma.$$

Proof. Assume that (ii) is not true. Then, for any $R > 0$, we get

$$\lim_{n \rightarrow \infty} \sup_{y \in \mathbb{R}^N} \int_{B_R(y)} |u_n|^2 dx = 0 = \lim_{n \rightarrow \infty} \sup_{y \in \mathbb{R}^N} \int_{B_R(y)} |v_n|^2 dx.$$

By using Lemma 2.1, we can deduce that

$$u_n, v_n \rightarrow 0 \text{ in } L^t(\mathbb{R}^N) \quad \forall t \in (2, 2_s^*).$$

This fact and (1.6), gives

$$\int_{\mathbb{R}^N} Q(u_n, v_n) dx \rightarrow 0. \quad (3.2)$$

Hence, by using $\langle \mathcal{J}'_0(u_n, v_n), (u_n, v_n) \rangle \rightarrow 0$, (1.5) and (3.2), we obtain

$$\|(u_n, v_n)\|_0^2 = \int_{\mathbb{R}^N} (Q_u(u_n, v_n)u_n + Q_v(u_n, v_n)v_n) dx = q \int_{\mathbb{R}^N} Q(u_n, v_n) dx = o_n(1),$$

which implies that (i) holds. \square

Theorem 3.1. *The problem (3.1) admits a weak solution.*

Proof. It is clear that \mathcal{J}_0 has a mountain-pass geometry, so, in view of Theorem 1.15 in [50], we can find a sequence $\{(u_n, v_n)\} \subset \mathbb{H}_0$ such that

$$\mathcal{J}_0(u_n, v_n) \rightarrow c_0 \text{ and } \mathcal{J}'_0(u_n, v_n) \rightarrow 0.$$

By using (1.5), we can see that

$$\begin{aligned} c_0 + o_n(1) \|(u_n, v_n)\|_0 &= \mathcal{J}_0(u_n, v_n) - \frac{1}{q} \langle \mathcal{J}'_0(u_n, v_n), (u_n, v_n) \rangle \\ &= \left(\frac{1}{2} - \frac{1}{q} \right) \|(u_n, v_n)\|_0^2, \end{aligned}$$

which implies that $\{(u_n, v_n)\}$ is bounded in \mathbb{H}_0 . As a consequence, in view of Theorem 2.1, we may assume that

$$\begin{aligned} (u_n, v_n) &\rightharpoonup (u, v) \text{ in } \mathbb{H}_0 \\ (u_n, v_n) &\rightarrow (u, v) \text{ in } L_{loc}^q(\mathbb{R}^N) \times L_{loc}^q(\mathbb{R}^N) \\ (u_n, v_n) &\rightarrow (u, v) \text{ a.e. in } \mathbb{R}^N. \end{aligned}$$

This fact and (Q2) allows us to deduce that $\mathcal{J}'_0(u, v) = 0$.

Now, we assume that $u \not\equiv 0$ and $v \not\equiv 0$. Then, by using (u^-, v^-) as test function, where $x^- = \max\{-x, 0\}$, and recalling that $(x - y)(x^- - y^-) \leq -|x^- - y^-|^2$ for any $x, y \in \mathbb{R}$, we can see that

$$\begin{aligned} 0 &= \langle \mathcal{J}'_0(u, v), (u^-, v^-) \rangle = \int_{\mathbb{R}^N} [(-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} u^- + (-\Delta)^{\frac{s}{2}} v (-\Delta)^{\frac{s}{2}} v^-] dx \\ &\quad + \int_{\mathbb{R}^N} (V_0 u u^- + W_0 v v^-) dx - \int_{\mathbb{R}^N} (Q_u(u, v) u^- + Q_v(u, v) v^-) dx \\ &= \iint_{\mathbb{R}^{2N}} \left[\frac{(u(x) - u(y))(u^-(x) - u^-(y))}{|x - y|^{N+2s}} + \frac{(v(x) - v(y))(v^-(x) - v^-(y))}{|x - y|^{N+2s}} \right] dx dy \\ &\quad + \int_{\mathbb{R}^N} (V_0 u u^- + W_0 v v^-) dx - \int_{\mathbb{R}^N} (Q_u(u, v) u^- + Q_v(u, v) v^-) dx \\ &\leq - \iint_{\mathbb{R}^{2N}} \left[\frac{|u^-(x) - u^-(y)|^2}{|x - y|^{N+2s}} + \frac{|v^-(x) - v^-(y)|^2}{|x - y|^{N+2s}} \right] dx dy \\ &\quad - \int_{\mathbb{R}^N} (V_0 (u^-)^2 + W_0 (v^-)^2) dx = -\|(u^-, v^-)\|_0^2, \end{aligned}$$

where we used the fact that $Q_u = 0$ on $(-\infty, 0) \times \mathbb{R}$ and $Q_v = 0$ on $\mathbb{R} \times (-\infty, 0)$.

As a consequence $u, v \geq 0$ in \mathbb{R}^N . Now, we know that ∇Q is $(q - 1)$ -homogeneous, so by using the conditions (Q4) and (Q5), and by applying the mean value theorem, we can deduce that $Q_u, Q_v \geq 0$. In view of (Q2), we can see that $z = u + v$ is a solution to $(-\Delta)^s z + V_0 z \leq C z^{p-1}$ in \mathbb{R}^N , for some constant $C > 0$. Hence, by using a Moser iteration argument (see for instance Proposition 5.1.1 in [23] or Theorem 1.2 in [8]) we can prove that $z \in L^\infty(\mathbb{R}^N)$, which implies that

$u, v \in L^\infty(\mathbb{R}^N)$. Then $Q_u(u, v)$ and $Q_v(u, v)$ are bounded, and by applying Proposition 2.9 in [46] we have $u, v \in C^{0,\alpha}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. From the Harnack inequality [12], we get $u, v > 0$ in \mathbb{R}^N .

At this point, we can show that $\mathcal{J}_0(u, v) = c_0$. Indeed, taking into account $(u, v) \in \mathcal{N}_0$, (1.5) and by using Fatou's Lemma, we get

$$\begin{aligned} c_0 &\leq \mathcal{J}_0(u, v) = \frac{q-2}{2} \int_{\mathbb{R}^N} Q(u, v) dx \\ &\leq \liminf_{n \rightarrow \infty} \frac{q-2}{2} \int_{\mathbb{R}^N} Q(u_n, v_n) dx \\ &= \liminf_{n \rightarrow \infty} \left[\mathcal{J}_0(u_n, v_n) - \frac{1}{2} \langle \mathcal{J}'_0(u_n, v_n), (u_n, v_n) \rangle \right] \\ &= c_0 \end{aligned}$$

which gives $\mathcal{J}_0(u, v) = c_0$.

Secondly, we consider the case $u \equiv 0$ or $v \equiv 0$. If $u \equiv 0$, we can use $\langle \mathcal{J}'_0(u, v), (u, v) \rangle = 0$ and (1.5) to see that

$$\|(0, v)\|_0^2 = \int_{\mathbb{R}^N} Q_u(0, v) v dx = q \int_{\mathbb{R}^N} Q(0, v) dx = 0$$

that is $v \equiv 0$. Analogously, we can prove that $v \equiv 0$ implies $u \equiv 0$. Therefore, if $u \equiv 0$ or $v \equiv 0$, we have $(u, v) = (0, 0)$.

Since $c_0 > 0$ and \mathcal{J}_0 is continuous, we can deduce that $\|(u_n, v_n)\|_0 \not\rightarrow 0$. Then, by using Lemma 3.1, we can find a sequence $(y_n) \subset \mathbb{R}^N$ and constants $R, \gamma > 0$ such that

$$\liminf_{n \rightarrow \infty} \int_{B_R(y_n)} (|u_n|^2 + |v_n|^2) dx \geq \gamma > 0. \quad (3.3)$$

Let us define $(\tilde{u}_n(x), \tilde{v}_n(x)) := (u_n(x + y_n), v_n(x + y_n))$. Then, by using the invariance of \mathbb{R}^N by translation, we can deduce that $\mathcal{J}_0(\tilde{u}_n, \tilde{v}_n) \rightarrow c_0$ and $\mathcal{J}'_0(\tilde{u}_n, \tilde{v}_n) \rightarrow 0$. Since (u_n, v_n) is bounded in \mathbb{H}_0 , we may assume that $(\tilde{u}_n, \tilde{v}_n) \rightharpoonup (\tilde{u}, \tilde{v})$ in \mathbb{H}_0 , $\tilde{u}_n \rightarrow \tilde{u}$ and $\tilde{v}_n \rightarrow \tilde{v}$ in $L^2_{loc}(\mathbb{R}^N)$, for some $(\tilde{u}, \tilde{v}) \in \mathbb{H}_0$ which is a critical point of \mathcal{J}_0 .

Thus, in view of (3.3), we have

$$\int_{B_R(0)} (|\tilde{u}|^2 + |\tilde{v}|^2) dx = \liminf_{n \rightarrow \infty} \int_{B_R(y_n)} (|u_n|^2 + |v_n|^2) dx \geq \gamma$$

which implies that $\tilde{u} \not\equiv 0$ or $\tilde{v} \not\equiv 0$. Arguing as before, we can deduce that both \tilde{u} and \tilde{v} are not identically zero. This ends the proof of theorem. \square

4. COMPACTNESS PROPERTIES

In this section we study the compactness properties of the functionals \mathcal{J}_ε . Firstly, we introduce some notations which we will use in the sequel.

If $\max\{V_\infty, W_\infty\} < \infty$, we define the functional $\mathcal{J}_\infty : \mathbb{H}_0 \rightarrow \mathbb{R}$ by setting

$$\mathcal{J}_\infty(u, v) = \frac{1}{2} \left(\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 + |(-\Delta)^{\frac{s}{2}} v|^2 dx + \int_{\mathbb{R}^N} (V_\infty u^2 + W_\infty v^2) dx \right) - \int_{\mathbb{R}^N} Q(u, v) dx,$$

and we denote by c_∞ the ground state level of \mathcal{J}_∞ , that is

$$c_\infty = \inf_{(u,v) \in \mathcal{N}_\infty} \mathcal{J}_\infty(u, v) = \inf_{(u,v) \in \mathbb{H}_0 \setminus \{(0,0)\}} \max_{t \geq 0} \mathcal{J}_\infty(tu, tv) > 0$$

where $\mathcal{N}_\infty = \{(u, v) \in \mathbb{H}_0 \setminus \{(0,0)\} : \mathcal{J}'_\infty(u, v)(u, v) = 0\}$. If $\max\{V_\infty, W_\infty\} = \infty$, we set $c_\infty = \infty$. Now, we prove the following useful lemmas which allows us to deduce a fundamental compactness result for \mathcal{J}_ε .

Lemma 4.1. *Suppose that $\max\{V_\infty, W_\infty\} < \infty$ and let $d \in \mathbb{R}$. Let $\{(u_n, v_n)\} \subset \mathbb{H}_0$ be a Palais-Smale sequence for \mathcal{J}_ε at the level d , such that $(u_n, v_n) \rightharpoonup (0, 0)$ in \mathbb{H}_ε . If $(u_n, v_n) \not\rightharpoonup (0, 0)$ in \mathbb{H}_ε , then $d \geq c_\infty$.*

Proof. Let $(t_n) \subset (0, \infty)$ be a sequence such that $(t_n u_n, t_n v_n) \in \mathcal{N}_\infty$. We begin proving the following claim:

Claim $t_0 = \limsup_{n \rightarrow \infty} t_n \leq 1$. Assume by contradiction that there exists $\lambda > 0$ such that

$$t_n \geq 1 + \lambda \text{ for any } n \in \mathbb{N}. \quad (4.1)$$

Since (u_n, v_n) is bounded in \mathbb{H}_ε , we get $\langle \mathcal{J}'_\varepsilon(u_n, v_n), (u_n, v_n) \rangle \rightarrow 0$, which together with (1.5) yields

$$\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u_n|^2 + |(-\Delta)^{\frac{s}{2}} v_n|^2 dx + \int_{\mathbb{R}^N} (V(\varepsilon x)|u_n|^2 + W(\varepsilon x)|v_n|^2) dx = q \int_{\mathbb{R}^N} Q(u_n, v_n) dx + o_n(1). \quad (4.2)$$

By using the fact that $(t_n u_n, t_n v_n) \in \mathcal{N}_\infty$, we get

$$t_n^2 \left(\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u_n|^2 + |(-\Delta)^{\frac{s}{2}} v_n|^2 dx + \int_{\mathbb{R}^N} (V_\infty |u_n|^2 + W_\infty |v_n|^2) dx \right) = q t_n^q \int_{\mathbb{R}^N} Q(u_n, v_n) dx. \quad (4.3)$$

Putting together (4.2) and (4.3), we obtain

$$q(t_n^{q-2} - 1) \int_{\mathbb{R}^N} Q(u_n, v_n) dx = \int_{\mathbb{R}^N} [(V_\infty - V(\varepsilon x))|u_n|^2 + (W_\infty - W(\varepsilon x))|v_n|^2] dx + o_n(1). \quad (4.4)$$

Now, by using (H2), we can see that for any $\eta > 0$ there exists $R > 0$ such that

$$V(\varepsilon x) \geq V_\infty - \eta, \quad W(\varepsilon x) \geq W_\infty - \eta \text{ for any } |x| \geq R. \quad (4.5)$$

On the other hand, in view of Theorem 2.1, we know that $u_n \rightarrow u$ and $v_n \rightarrow v$ in $L_{loc}^t(\mathbb{R}^N)$ for any $t \in [2, 2_s^*)$.

Taking into account this fact, $\|(u_n, v_n)\|_\varepsilon \leq C$, (4.4) and (4.5), we have

$$q((1 + \lambda)^{q-2} - 1) \int_{\mathbb{R}^N} Q(u_n, v_n) dx \leq q(t_n^{q-2} - 1) \int_{\mathbb{R}^N} Q(u_n, v_n) dx \leq C' \eta + o_n(1). \quad (4.6)$$

Since $\|(u_n, v_n)\|_\varepsilon \not\rightarrow 0$, we can proceed as in the proof of Lemma 3.1 to deduce that there exist a sequence $(y_n) \subset \mathbb{R}^N$ and constants $R, \gamma > 0$ such that

$$\int_{B_R(y_n)} (|u_n|^2 + |v_n|^2) dx \geq \gamma > 0. \quad (4.7)$$

Let us define $(\tilde{u}_n(x), \tilde{v}_n(x)) = (u_n(x + y_n), v_n(x + y_n))$. Then, we may assume that $(\tilde{u}_n, \tilde{v}_n) \rightharpoonup (u, v)$ in \mathbb{H}_ε for some nonnegative functions u and v such that $\mathcal{J}'_\varepsilon(u, v) = 0$. From (4.7), it is easy to see that $u \not\equiv 0$ or $v \not\equiv 0$. Moreover, arguing as in the proof of Theorem 3.1, we deduce that u and v are positive in \mathbb{R}^N . Then, by using Fatou's Lemma and (4.6), we get

$$0 < q((1 + \lambda)^{q-2} - 1) \int_{\mathbb{R}^N} Q(u, v) dx \leq C' \eta$$

for any $\eta > 0$, and this gives a contradiction. Therefore, we can infer that $t_0 \leq 1$.

Now, it is convenient to distinguish the following cases.

Case 1 $t_0 < 1$. Then, we may assume that $t_n < 1$ for all $n \in \mathbb{N}$.

By using (1.5), we can see that

$$\begin{aligned}
c_\infty &\leq \mathcal{J}_\infty(t_n u_n, t_n v_n) = \mathcal{J}_\infty(t_n u_n, t_n v_n) - \frac{1}{2} \langle \mathcal{J}'_\infty(t_n u_n, t_n v_n), (t_n u_n, t_n v_n) \rangle \\
&= t_n^q \left(\frac{q-2}{2} \right) \int_{\mathbb{R}^N} Q(u_n, v_n) dx \\
&\leq \left(\frac{q-2}{2} \right) \int_{\mathbb{R}^N} Q(u_n, v_n) dx \\
&= \mathcal{J}_\infty(t_n u_n, t_n v_n) - \frac{1}{2} \langle \mathcal{J}'_\infty(u_n, v_n), (u_n, v_n) \rangle \\
&= d + o_n(1)
\end{aligned}$$

so we deduce that $d \geq c_\infty$.

Case 2 $t_0 = 1$. Up to a subsequence, we may assume that $t_n \rightarrow 1$. Moreover, we have

$$d + o_n(1) \geq c_\infty + \mathcal{J}_\varepsilon(u_n, v_n) - \mathcal{J}_\infty(t_n u_n, t_n v_n). \quad (4.8)$$

Now, fix $\eta > 0$. Taking into account (4.5), q -homogeneity of Q , the boundedness of $\{(u_n, v_n)\}$ and $t_n \rightarrow 1$, we can see that

$$\begin{aligned}
\mathcal{J}_\varepsilon(u_n, v_n) - \mathcal{J}_\infty(t_n u_n, t_n v_n) &= \frac{(1-t_n^2)}{2} \left(\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u_n|^2 + |(-\Delta)^{\frac{s}{2}} v_n|^2 dx \right) \\
&\quad + \frac{1}{2} \int_{\mathbb{R}^N} V(\varepsilon x) |u_n|^2 + W(\varepsilon x) |v_n|^2 dx \\
&\quad - \frac{t_n^2}{2} \int_{\mathbb{R}^N} (V_\infty |u_n|^2 + W_\infty |v_n|^2) dx + (t_n^q - 1) \int_{\mathbb{R}^N} Q(u_n, v_n) dx \\
&\geq o_n(1) - C\eta.
\end{aligned} \quad (4.9)$$

Putting together (4.8) and (4.9), and by using the arbitrariness of η , we conclude that $d \geq c_\infty$. \square

Lemma 4.2. Assume that $\max\{V_\infty, W_\infty\} = \infty$. Let $\{(u_n, v_n)\} \subset \mathbb{H}_0$ be a Palais-Smale sequence for \mathcal{J}_ε at the level d , such that $(u_n, v_n) \rightharpoonup (0, 0)$ in \mathbb{H}_ε . Then $(u_n, v_n) \rightarrow (0, 0)$ in \mathbb{H}_ε .

Proof. For any $(a, b) \in \mathbb{R}_+^2$, we define

$$c_{(a,b)} = \inf_{(u,v) \in \mathbb{H}_0 \setminus \{(0,0)\}} \max_{t \geq 0} \mathcal{J}_{(a,b)}(tu, tv)$$

where

$$\mathcal{J}_{(a,b)}(u, v) = \frac{1}{2} \left(\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 + |(-\Delta)^{\frac{s}{2}} v|^2 dx \right) + \frac{1}{2} \int_{\mathbb{R}^N} (a|u|^2 + b|v|^2) dx - \int_{\mathbb{R}^N} Q(u, v) dx.$$

We note that if $a > a'$ then $c_{(a,b)} > c_{(a',b)}$, and that $\lim_{a^2+b^2 \rightarrow \infty} c_{(a,b)} = \infty$.

Now, fixed $(a, b) \in \mathbb{R}_+^2$, we can proceed as in the proof of Theorem 3.1, to see that $c_{(a,b)}$ is achieved in some couple (u, v) with u and v are positive functions in \mathbb{R}^N .

Since $\max\{V_\infty, W_\infty\} = \infty$, we can take $(a, b) \in \mathbb{R}_+^2$ such that $c_{(a,b)} > d$, and, for any fixed $\eta > 0$, there exists $R > 0$ such that

$$V(\varepsilon x) \geq a - \eta, \quad W(\varepsilon x) \geq b - \eta \text{ for any } |x| \geq R. \quad (4.10)$$

We observe that if $W_\infty < \infty$, we can choose $b = W_\infty$ and $a > 0$ large, and when $V_\infty = W_\infty = \infty$, we take both a and b sufficiently large.

If by contradiction $(u_n, v_n) \rightharpoonup (0, 0)$ in \mathbb{H}_ε , we proceed as in the proof of Lemma 4.1 and using (4.10) we deduce that $d \geq c_{(a,b)}$. But this is impossible, because we chose (a, b) such that $c_{(a,b)} > d$. Therefore, we get $(u_n, v_n) \rightarrow (0, 0)$ in \mathbb{H}_ε . \square

Now, we are ready to give the proof of the following compactness result.

Theorem 4.1. *The functional \mathcal{J}_ε constrained to \mathcal{N}_ε satisfies the Palais-Smale condition at every level $d < c_\infty$.*

Proof. Let $\{(u_n, v_n)\} \subset \mathcal{N}_\varepsilon$ be a sequence such that $\mathcal{J}_\varepsilon(u_n, v_n) \rightarrow d$ and $\|\mathcal{J}'_\varepsilon(u_n, v_n)|_{\mathcal{N}_\varepsilon}\|_* \rightarrow 0$. Then, there exists $(\lambda_n) \subset \mathbb{R}$ such that

$$\mathcal{J}'_\varepsilon(u_n, v_n) = \lambda_n \mathcal{I}'_\varepsilon(u_n, v_n) + o_n(1),$$

where

$$\mathcal{I}_\varepsilon(u, v) := \|(u, v)\|_\varepsilon^2 - q \int_{\mathbb{R}^N} Q(u, v) dx.$$

Hence

$$0 = \langle \mathcal{J}'_\varepsilon(u_n, v_n), (u_n, v_n) \rangle = \lambda_n \langle \mathcal{I}'_\varepsilon(u_n, v_n), (u_n, v_n) \rangle + o_n(1) = \lambda_n(2 - q) \|(u_n, v_n)\|_\varepsilon^2 + o_n(1),$$

and by using (2.3), we deduce that $\lambda_n \rightarrow 0$. Then $\mathcal{J}'_\varepsilon(u_n, v_n) \rightarrow 0$ in the dual of \mathbb{H}_ε .

Since the Palais-Smale of \mathcal{J}_ε are bounded, we may assume that $(u_n, v_n) \rightharpoonup (u, v)$ in \mathbb{H}_ε , for some (u, v) which is a critical point of \mathcal{J}_ε .

Now, we set $(w_n, z_n) := (u_n - u, v_n - v)$. By using the weak convergence of $\{(u_n, v_n)\}$ and (1.6), we can apply the Brezis-Lieb Lemma and the splitting Lemma (we recall that Q has subcritical growth), to deduce that

$$\begin{aligned} \mathcal{J}_\varepsilon(w_n, z_n) &= \mathcal{J}_\varepsilon(u_n, v_n) - \mathcal{J}_\varepsilon(u, v) + o_n(1) \\ &= d - \mathcal{J}_\varepsilon(u, v) + o_n(1) =: \tilde{d} + o_n(1) \end{aligned}$$

and

$$\mathcal{J}'_\varepsilon(w_n, z_n) = o_n(1).$$

Since $\mathcal{J}'_\varepsilon(u, v) = 0$, we can see that

$$\mathcal{J}_\varepsilon(u, v) = \mathcal{J}_\varepsilon(u, v) - \frac{1}{2} \langle \mathcal{J}'_\varepsilon(u, v), (u, v) \rangle = \frac{q-2}{2} \int_{\mathbb{R}^N} Q(u, v) dx \geq 0,$$

which implies that $\tilde{d} < c_\infty$.

Now, if we assume that $\max\{V_\infty, W_\infty\} < \infty$, from Lemma 4.1 we deduce that $(w_n, z_n) \rightarrow (0, 0)$ in \mathbb{H}_ε , that is $(u_n, v_n) \rightarrow (u, v)$ in \mathbb{H}_ε . In the case $\max\{V_\infty, W_\infty\} = \infty$, we can apply Lemma 4.2 to deduce that $(u_n, v_n) \rightarrow (u, v)$ in \mathbb{H}_ε . □

Arguing as in the above theorem, it is easy to prove that it holds the following result.

Corollary 4.1. *The critical points of \mathcal{J}_ε constrained to \mathcal{N}_ε are critical points of \mathcal{J}_ε in \mathbb{H}_ε*

5. BARYCENTER MAP AND MULTIPLICITY OF SOLUTIONS TO (2.2)

In this section, our main purpose is to apply the Ljusternik-Schnirelmann category theory to prove a multiplicity result for system (2.2). In order to obtain our main result, first we give some useful lemmas. We start proving the following result.

Lemma 5.1. *Let $\varepsilon_n \rightarrow 0^+$ and $\{(u_n, v_n)\} \subset \mathcal{N}_{\varepsilon_n}$ be such that $\mathcal{J}_{\varepsilon_n}(u_n, v_n) \rightarrow c_0$. Then there exists $\{\tilde{y}_n\} \subset \mathbb{R}^N$ such that the translated sequence*

$$(\tilde{u}_n(x), \tilde{v}_n(x)) := (u_n(x + \tilde{y}_n), v_n(x + \tilde{y}_n))$$

has a subsequence which converges in \mathbb{H}_0 . Moreover, up to a subsequence, $\{y_n\} := \{\varepsilon_n \tilde{y}_n\}$ is such that $y_n \rightarrow y \in M$.

Proof. Since $\langle \mathcal{J}'_{\varepsilon_n}(u_n, v_n), (u_n, v_n) \rangle = 0$ and $\mathcal{J}_{\varepsilon_n}(u_n, v_n) \rightarrow c_0$, we can argue as in the proof of Proposition 3.1 to deduce that $\{(u_n, v_n)\}$ is bounded. Let us observe that $\|(u_n, v_n)\| \not\rightarrow 0$ since $c_0 > 0$. Therefore, as in the proof of Lemma 3.1, we can find a sequence $\{\tilde{y}_n\} \subset \mathbb{R}^N$ and constants $R, \gamma > 0$ such that

$$\liminf_{n \rightarrow \infty} \int_{B_R(y_n)} (|u_n|^2 + |v_n|^2) dx \geq \gamma,$$

which implies that

$$(\tilde{u}_n, \tilde{v}_n) \rightharpoonup (\tilde{u}, \tilde{v}) \text{ weakly in } \mathbb{H}_0,$$

where $(\tilde{u}_n(x), \tilde{v}_n(x)) := (u_n(x + \tilde{y}_n), v_n(x + \tilde{y}_n))$ and $(\tilde{u}, \tilde{v}) \neq (0, 0)$.

Let $\{t_n\} \subset (0, +\infty)$ be such that $(\hat{u}_n, \hat{v}_n) := (t_n \tilde{u}_n, t_n \tilde{v}_n) \in \mathcal{N}_0$, and set $y_n := \varepsilon_n \tilde{y}_n$.

By using the change of variables $z \mapsto x + \tilde{y}_n$, we can see that

$$\begin{aligned} \mathcal{J}_0(\hat{u}_n, \hat{v}_n) &\leq \frac{t_n^2}{2} \left(\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} \tilde{u}_n|^2 + |(-\Delta)^{\frac{s}{2}} \tilde{v}_n|^2 dx \right) - \int_{\mathbb{R}^N} Q(t_n \tilde{u}_n, t_n \tilde{v}_n) dx \\ &\quad + \frac{t_n^2}{2} \int_{\mathbb{R}^N} (V(\varepsilon_n(x + \tilde{y}_n)) |\tilde{u}_n|^2 + W(\varepsilon_n(x + \tilde{y}_n)) |\tilde{v}_n|^2) dx \\ &= \mathcal{J}_{\varepsilon_n}(t_n u_n, t_n v_n) \leq \mathcal{J}_{\varepsilon_n}(u_n, v_n) = c_0 + o_n(1). \end{aligned}$$

Taking into account that $c_0 \leq \mathcal{J}_0(\hat{u}_n, \hat{v}_n)$, we can infer $\mathcal{J}_0(\hat{u}_n, \hat{v}_n) \rightarrow c_0$.

Now, the sequence $\{t_n\}$ is bounded since $\{(\tilde{u}_n, \tilde{v}_n)\}$ and $\{(\hat{u}_n, \hat{v}_n)\}$ are bounded and $(\tilde{u}_n, \tilde{v}_n) \not\rightarrow 0$. Therefore, up to a subsequence, $t_n \rightarrow t_0 \geq 0$. Indeed $t_0 > 0$. Otherwise, if $t_0 = 0$, from the boundedness of $\{(\tilde{u}_n, \tilde{v}_n)\}$, we get $(\hat{u}_n, \hat{v}_n) = t_n(\tilde{u}_n, \tilde{v}_n) \rightarrow (0, 0)$, that is $\mathcal{J}_0(\hat{u}_n, \hat{v}_n) \rightarrow 0$ in contrast with the fact $c_0 > 0$. Thus $t_0 > 0$, and up to a subsequence, we have $(\hat{u}_n, \hat{v}_n) \rightharpoonup t_0(\tilde{u}, \tilde{v}) = (\hat{u}, \hat{v})$ weakly in \mathbb{H}_0 . Hence, it holds

$$\mathcal{J}_0(\hat{u}_n, \hat{v}_n) \rightarrow c_0 \quad \text{and} \quad (\hat{u}_n, \hat{v}_n) \rightharpoonup (\hat{u}, \hat{v}) \text{ weakly in } \mathbb{H}_0.$$

From Theorem 3.1, we deduce that $(\hat{u}_n, \hat{v}_n) \rightarrow (\hat{u}, \hat{v})$ in \mathbb{H}_0 , that is $(\tilde{u}_n, \tilde{v}_n) \rightarrow (\tilde{u}, \tilde{v})$ in \mathbb{H}_0 .

Now, we show that $\{y_n\}$ has a subsequence such that $y_n \rightarrow y \in M$. Assume by contradiction that $\{y_n\}$ is not bounded, that is there exists a subsequence, still denoted by $\{y_n\}$, such that $|y_n| \rightarrow +\infty$. Firstly, we deal with the case $\max\{V_\infty, W_\infty\} = \infty$.

Since $(u_n, v_n) \in \mathcal{N}_{\varepsilon_n}$, we can see that

$$q \int_{\mathbb{R}^N} Q(\tilde{u}_n, \tilde{v}_n) dx \geq \int_{\mathbb{R}^N} V(\varepsilon_n x + y_n) |\tilde{u}_n|^2 dx + \int_{\mathbb{R}^N} W(\varepsilon_n x + y_n) |\tilde{v}_n|^2 dx.$$

By applying Fatou's Lemma, we deduce that

$$\liminf_{n \rightarrow \infty} \int_{\mathbb{R}^N} Q(\tilde{u}_n, \tilde{v}_n) dx = \infty, \tag{5.1}$$

which is impossible because the boundedness of $\{(u_n, v_n)\}$ and (1.6) yield

$$\left| \int_{\mathbb{R}^N} Q(\tilde{u}_n, \tilde{v}_n) dx \right| \leq C \text{ for any } n \in \mathbb{N}.$$

Let us consider the case $\max\{V_\infty, W_\infty\} < \infty$.

Since $(\hat{u}_n, \hat{v}_n) \rightarrow (\hat{u}, \hat{v})$ strongly in \mathbb{H}_0 and $V_0 < \max\{V_\infty, W_\infty\}$, we have

$$\begin{aligned} c_0 &= \mathcal{J}_0(\hat{u}, \hat{v}) < \mathcal{J}_\infty(\hat{u}, \hat{v}) \\ &\leq \liminf_{n \rightarrow \infty} \left\{ \frac{1}{2} \left(\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} \hat{u}_n|^2 + |(-\Delta)^{\frac{s}{2}} \hat{v}_n|^2 dx \right) - \int_{\mathbb{R}^N} Q(\hat{u}_n, \hat{v}_n) dx \right. \\ &\quad \left. + \frac{1}{2} \int_{\mathbb{R}^N} (V(\varepsilon_n x + y_n) |\hat{u}_n|^2 + W(\varepsilon_n x + y_n) |\hat{v}_n|^2) dx \right\} \\ &= \liminf_{n \rightarrow \infty} \mathcal{J}_{\varepsilon_n}(t_n u_n, t_n v_n) \leq \liminf_{n \rightarrow \infty} \mathcal{J}_{\varepsilon_n}(u_n, v_n) = c_0 \end{aligned} \tag{5.2}$$

which gives a contradiction.

Thus $\{y_n\}$ is bounded and, up to a subsequence, we may assume that $y_n \rightarrow y$. If $y \notin M$, then $V_0 < \max\{V(y), W(y)\}$ and we have

$$c_0 = \mathcal{J}_0(\hat{u}, \hat{v}) < \frac{1}{2} \left(\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} \hat{u}|^2 + |(-\Delta)^{\frac{s}{2}} \hat{v}|^2 dx \right) + \frac{1}{2} \int_{\mathbb{R}^N} (V(y)|\hat{u}|^2 + W(y)|\hat{v}|^2) dx - \int_{\mathbb{R}^N} Q(\hat{u}, \hat{v}) dx.$$

Repeating the same argument in (5.2), we get a contradiction. Therefore, we can conclude that $y \in M$. \square

For any $\delta > 0$, we set

$$M_\delta = \{x \in \mathbb{R}^N : \text{dist}(x, M) \leq \delta\}.$$

Let $(w_1, w_2) \in \mathbb{H}_0$ be a solution for (3.1) (which there exists in view of Theorem 3.1), and we define

$$\Psi_{i,\varepsilon,z}(x) = \eta(|\varepsilon x - z|) w_i \left(\frac{\varepsilon x - z}{\varepsilon} \right) \quad i = 1, 2.$$

where $\eta \in C_0^\infty(\mathbb{R}_+, [0, 1])$ is a function satisfying $\eta(t) = 1$ if $0 \leq t \leq \frac{\delta}{2}$ and $\eta(t) = 0$ if $t \geq \delta$. Let $t_\varepsilon > 0$ be the unique positive number such that

$$\max_{t \geq 0} \mathcal{J}_\varepsilon(t\Psi_{1,\varepsilon,z}, t\Psi_{1,\varepsilon,z}) = \mathcal{J}_\varepsilon(t_\varepsilon\Psi_{2,\varepsilon,z}, t_\varepsilon\Psi_{1,\varepsilon,z}).$$

Finally, we consider $\Phi_\varepsilon(z) = (t_\varepsilon\Psi_{1,\varepsilon,z}, t_\varepsilon\Psi_{2,\varepsilon,z})$. Since $\mathcal{J}_0(w_1, w_2) = c_0$ and M is compact, we can prove the following result.

Lemma 5.2. *The functional Φ_ε satisfies the following limit*

$$\lim_{\varepsilon \rightarrow 0} \mathcal{J}_\varepsilon(\Phi_\varepsilon(y)) = c_0 \text{ uniformly in } y \in M. \quad (5.3)$$

Proof. Assume by contradiction that there there exists $\delta_0 > 0$, $(y_n) \subset M$ and $\varepsilon_n \rightarrow 0$ such that

$$|\mathcal{J}_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) - c_0| \geq \delta_0. \quad (5.4)$$

We first show that $\lim_{n \rightarrow \infty} t_{\varepsilon_n} < \infty$. Let us observe that by using the change of variable $z = \frac{\varepsilon_n x - y_n}{\varepsilon_n}$, if $z \in B_{\frac{\delta}{\varepsilon_n}}(0)$, it follows that $\varepsilon_n z \in B_\delta(0)$ and $\varepsilon_n x + y_n \in B_\delta(y_n) \subset M_\delta$.

Then we have

$$\begin{aligned} \mathcal{J}_\varepsilon(\Phi_{\varepsilon_n}(y_n)) &= \frac{t_{\varepsilon_n}^2}{2} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}(\eta(|\varepsilon_n z| w_1(z)))|^2 dz + \frac{t_{\varepsilon_n}^2}{2} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}(\eta(|\varepsilon_n z| w_2(z)))|^2 dz \\ &\quad + \frac{t_{\varepsilon_n}^2}{2} \int_{\mathbb{R}^N} V(\varepsilon_n z + y_n)(\eta(|\varepsilon_n z| w_1(z)))^2 dz + \frac{t_{\varepsilon_n}^2}{2} \int_{\mathbb{R}^N} W(\varepsilon_n z + y_n)(\eta(|\varepsilon_n z| w_2(z)))^2 dz \\ &\quad - \int_{\mathbb{R}^N} Q(t_{\varepsilon_n} \eta(|\varepsilon_n z|) w_1(z), t_{\varepsilon_n} \eta(|\varepsilon_n z|) w_2(z)) dz. \end{aligned} \quad (5.5)$$

Now, let assume that $t_{\varepsilon_n} \rightarrow \infty$. From the definition of t_{ε_n} , (Q1) and (1.5), we get

$$\|(\Psi_{1,\varepsilon_n,y_n}, \Psi_{2,\varepsilon_n,y_n})\|_{\varepsilon_n}^2 = q t_{\varepsilon_n}^{q-2} \int_{\mathbb{R}^N} Q(\eta(|\varepsilon_n z|) w_1(z), \eta(|\varepsilon_n z|) w_2(z)) dz \quad (5.6)$$

Since $\eta = 1$ in $B_{\frac{\delta}{2}}(0)$ and $B_{\frac{\delta}{2}}(0) \subset B_{\frac{\delta}{2\varepsilon_n}}(0)$ for n big enough, and w_1, w_2 are continuous and positive in \mathbb{R}^N (see proof of Theorem 3.1) we obtain

$$\|(\Psi_{1,\varepsilon_n,y_n}, \Psi_{2,\varepsilon_n,y_n})\|_{\varepsilon_n}^2 \geq q t_{\varepsilon_n}^{q-2} \int_{B_{\frac{\delta}{2}}(0)} Q(w_1(z), w_2(z)) dz \geq C_{\delta,q} t_{\varepsilon_n}^{q-2}, \quad (5.7)$$

where $C_{\delta,q} = q \left(\frac{\delta}{2}\right)^N \omega_N \min_{z \in \bar{B}_{\frac{\delta}{2}}(0)} Q(w_1(z), w_2(z)) > 0$. Taking the limit as $n \rightarrow \infty$ in (5.7) we can deduce that

$$\lim_{n \rightarrow \infty} \|(\Psi_{1,\varepsilon_n,y_n}, \Psi_{2,\varepsilon_n,y_n})\|_{\varepsilon_n}^2 = \infty$$

which is a contradiction because of

$$\lim_{n \rightarrow \infty} \|(\Psi_{1,\varepsilon_n,y_n}, \Psi_{2,\varepsilon_n,y_n})\|_{\varepsilon_n}^2 = \|(w_1, w_2)\|_0^2 \in (0, \infty)$$

in view of the Dominated Convergence Theorem.

Thus, (t_{ε_n}) is bounded, and we can assume that $t_{\varepsilon_n} \rightarrow t_0 \geq 0$. Clearly, if $t_0 = 0$, by limitation of $\|(\Psi_{1,\varepsilon_n,y_n}, \Psi_{2,\varepsilon_n,y_n})\|_{\varepsilon_n}^2$, the growth assumptions on Q , and (5.6), we can deduce that $\|(\Psi_{1,\varepsilon_n,y_n}, \Psi_{2,\varepsilon_n,y_n})\|_{\varepsilon_n}^2 \rightarrow 0$, which is impossible. Hence, $t_0 > 0$.

Now, by using the Dominated Convergence Theorem, we can see that as $n \rightarrow \infty$

$$\int_{\mathbb{R}^N} Q(\Psi_{1,\varepsilon_n,y_n}, \Psi_{2,\varepsilon_n,y_n}) dx \rightarrow \int_{\mathbb{R}^N} Q(w_1, w_2) dx.$$

Then, taking the limit as $n \rightarrow \infty$ in (5.6), we obtain

$$\|(w_1, w_2)\|_0^2 = qt_0^{q-2} \int_{\mathbb{R}^N} Q(w_1, w_2) dx.$$

By using the fact that $(w_1, w_2) \in \mathcal{N}_0$, we deduce that $t_0 = 1$. Moreover, from (5.5), we have

$$\lim_{n \rightarrow \infty} \mathcal{J}_\varepsilon(\Phi_{\varepsilon_n}(y_n)) = \mathcal{J}_0(w_1, w_2) = c_0,$$

which contradicts (5.4). □

Now, we are in the position to define the barycenter map. We take $\rho > 0$ such that $M_\delta \subset B_\rho$, and we consider $\Upsilon : \mathbb{R}^N \rightarrow \mathbb{R}^N$ defined by setting

$$\Upsilon(x) = \begin{cases} x & \text{if } |x| < \rho \\ \frac{\rho x}{|x|} & \text{if } |x| \geq \rho. \end{cases}$$

We define the barycenter map $\beta_\varepsilon : \mathcal{N}_\varepsilon \rightarrow \mathbb{R}^N$ as follows

$$\beta_\varepsilon(u, v) = \frac{\int_{\mathbb{R}^N} \Upsilon(\varepsilon x)(u^2(x) + v^2(x)) dx}{\int_{\mathbb{R}^N} u^2(x) + v^2(x) dx}.$$

Lemma 5.3. *The functional Φ_ε verifies the following limit*

$$\lim_{\varepsilon \rightarrow 0} \beta_\varepsilon(\Phi_\varepsilon(y)) = y \text{ uniformly in } y \in M. \quad (5.8)$$

Proof. Suppose by contradiction that there exists $\delta_0 > 0$, $(y_n) \subset M$ and $\varepsilon_n \rightarrow 0$ such that

$$|\beta_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) - y_n| \geq \delta_0. \quad (5.9)$$

By using the definitions of $\Phi_{\varepsilon_n}(z_n)$, β_{ε_n} , η and the change of variable $x \mapsto \frac{\varepsilon_n x - y_n}{\varepsilon_n}$, we can see that

$$\beta_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) = y_n + \frac{\int_{\mathbb{R}^N} [\Upsilon(\varepsilon_n x + y_n) - y_n] |\eta(\varepsilon_n x)|^2 (|w_1(x)|^2 + |w_2(x)|^2) dx}{\int_{\mathbb{R}^N} |\eta(\varepsilon_n x)|^2 (|w_1(x)|^2 + |w_2(x)|^2) dx}.$$

Taking into account $(y_n) \subset M \subset B_\rho$ and the Dominated Convergence Theorem, we can infer that

$$|\beta_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) - y_n| = o_n(1)$$

which contradicts (5.9). □

At this point, we introduce a subset $\tilde{\mathcal{N}}_\varepsilon$ of \mathcal{N}_ε by taking a function $h : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that $h(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$, and setting

$$\tilde{\mathcal{N}}_\varepsilon = \{(u, v) \in \mathcal{N}_\varepsilon : \mathcal{J}_\varepsilon(u) \leq c_0 + h(\varepsilon)\}.$$

Fixed $y \in M$, we conclude from Lemma 5.2 that $h(\varepsilon) = |\mathcal{J}_\varepsilon(\Phi_\varepsilon(y)) - c_0| \rightarrow 0$ as $\varepsilon \rightarrow 0$. Hence $\Phi_\varepsilon(y) \in \tilde{\mathcal{N}}_\varepsilon$, and $\tilde{\mathcal{N}}_\varepsilon \neq \emptyset$ for any $\varepsilon > 0$. Moreover, we have the following lemma.

Lemma 5.4.

$$\lim_{\varepsilon \rightarrow 0} \sup_{(u,v) \in \tilde{\mathcal{N}}_\varepsilon} \text{dist}(\beta_\varepsilon(u, v), M_\delta) = 0.$$

Proof. Let $\varepsilon_n \rightarrow 0$ as $n \rightarrow \infty$. For any $n \in \mathbb{N}$, there exists $(u_n, v_n) \in \tilde{\mathcal{N}}_{\varepsilon_n}$ such that

$$\sup_{(u,v) \in \tilde{\mathcal{N}}_{\varepsilon_n}} \inf_{y \in M_\delta} |\beta_{\varepsilon_n}(u, v) - y| = \inf_{y \in M_\delta} |\beta_{\varepsilon_n}(u_n, v_n) - y| + o_n(1).$$

Therefore, it suffices to prove that there exists $(y_n) \subset M_\delta$ such that

$$\lim_{n \rightarrow \infty} |\beta_{\varepsilon_n}(u_n, v_n) - y_n| = 0. \quad (5.10)$$

We note that $\{(u_n, v_n)\} \subset \tilde{\mathcal{N}}_{\varepsilon_n} \subset \mathcal{N}_{\varepsilon_n}$, from which we deduce that

$$c_0 \leq c_{\varepsilon_n} \leq \mathcal{J}_{\varepsilon_n}(u_n, v_n) \leq c_0 + h(\varepsilon_n).$$

This yields $\mathcal{J}_{\varepsilon_n}(u_n, v_n) \rightarrow c_0$. By using Lemma 5.1, there exists $(\tilde{y}_n) \subset \mathbb{R}^N$ such that $y_n = \varepsilon_n \tilde{y}_n \in M_\delta$ for n sufficiently large. By setting $(\tilde{u}_n(x), \tilde{v}_n(x)) = (u_n(\cdot + \tilde{y}_n), v_n(\cdot + \tilde{y}_n))$, we can see that

$$\beta_{\varepsilon_n}(u_n, v_n) = y_n + \frac{\int_{\mathbb{R}^N} [\Upsilon(\varepsilon_n x + y_n) - y_n](\tilde{u}_n^2 + \tilde{v}_n^2) dx}{\int_{\mathbb{R}^N} (\tilde{u}_n^2 + \tilde{v}_n^2) dx}.$$

Since $(\tilde{u}_n, \tilde{v}_n) \rightarrow (u, v)$ in \mathbb{H}_0 and $\varepsilon_n x + y_n \rightarrow y \in M$, we deduce that $\beta_{\varepsilon_n}(u_n, v_n) = y_n + o_n(1)$, that is (5.10) holds. \square

Now, we are ready to present the proof of the first multiplicity result related to (1.7).

Proof of thm 1.1. Given $\delta > 0$, we can apply Lemma 5.2, Lemma 5.3 and Lemma 5.4 to find some $\varepsilon_\delta > 0$ such that for any $\varepsilon \in (0, \varepsilon_\delta)$, the diagram

$$M \xrightarrow{\Phi_\varepsilon} \tilde{\mathcal{N}}_\varepsilon \xrightarrow{\beta_\varepsilon} M_\delta$$

is well-defined and $\beta_\varepsilon \circ \Phi_\varepsilon$ is homotopically equivalent to the map $\iota : M \rightarrow M_\delta$. By using the definition of $\tilde{\mathcal{N}}_\varepsilon$ and taking ε_δ sufficiently small, we may assume that \mathcal{J}_ε verifies the Palais-Smale condition in $\tilde{\mathcal{N}}_\varepsilon$. Therefore, standard Ljusternik-Schnirelmann theory [50] provides at least $\text{cat}_{\tilde{\mathcal{N}}_\varepsilon}(\tilde{\mathcal{N}}_\varepsilon)$ critical points (u_i, v_i) of \mathcal{J}_ε restricted to \mathcal{N}_ε . Using the arguments in [11], we can see that $\text{cat}_{\tilde{\mathcal{N}}_\varepsilon}(\tilde{\mathcal{N}}_\varepsilon) \geq \text{cat}_{M_\delta}(M)$. From Corollary 4.1 and the arguments contained in the proof of Theorem 3.1, we can conclude that $u_i > 0$, $v_i > 0$ and (u_i, v_i) is a solution to (2.2). \square

6. PROOF OF THEOREM 1.2

In this last section we deal with the nonlocal system in the critical case. As in the Section 3, we consider the following autonomous critical system

$$\begin{cases} (-\Delta)^s u + V_0 u = Q_u(u, v) + \frac{\alpha}{\alpha+\beta} |u|^{\alpha-2} u |v|^\beta & \text{in } \mathbb{R}^N \\ (-\Delta)^s v + W_0 v = Q_v(u, v) + \frac{\beta}{\alpha+\beta} |u|^\alpha |v|^{\beta-2} v & \text{in } \mathbb{R}^N \\ u, v > 0 & \text{in } \mathbb{R}^N, \end{cases} \quad (6.1)$$

and let us define the following functional

$$\mathcal{J}_0(u, v) = \frac{1}{2} \|(u, v)\|_0^2 - \int_{\mathbb{R}^N} Q(u, v) dx - \frac{1}{\alpha + \beta} \int_{\mathbb{R}^N} (u^+)^{\alpha} (v^+)^{\beta} dx.$$

and its ground state level

$$m_0 = \inf_{(u, v) \in \mathcal{N}_0} \mathcal{J}_0(u, v) = \inf_{(u, v) \in X_0 \setminus \{(0, 0)\}} \max_{t \geq 0} \mathcal{J}_0(tu, tv) > 0.$$

Now, we denote by

$$\tilde{S}_* = \tilde{S}_*(\alpha, \beta) = \inf_{u, v \in H^s(\mathbb{R}^N) \setminus \{(0, 0)\}} \frac{\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 + |(-\Delta)^{\frac{s}{2}} v|^2 dx}{\left(\int_{\mathbb{R}^N} |u|^{\alpha} |v|^{\beta} dx \right)^{\frac{2}{2_s^*}}}. \quad (6.2)$$

In the next lemma, we prove an interesting relation between S_* and \tilde{S}_* .

Lemma 6.1. *It holds*

$$\tilde{S}_* = S_* \left[\left(\frac{\alpha}{\beta} \right)^{\frac{\beta}{2_s^*}} + \left(\frac{\beta}{\alpha} \right)^{\frac{\alpha}{2_s^*}} \right].$$

Moreover, if w realizes S_* , then (Aw, Bw) realizes \tilde{S}_* where A and B are such that $\frac{A}{B} = \sqrt{\frac{\alpha}{\beta}}$.

Proof. Let (w_n) be a minimizing sequence for S_* . Let p and q two positive numbers which will be chosen later. Choosing $u_n = pw_n$ and $v_n = qw_n$ in the quotient (6.2), we have

$$\frac{p^2 + q^2}{(p^{\alpha} q^{\beta})^{\frac{2}{2_s^*}}} \frac{\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} w_n|^2 dx}{\left(\int_{\mathbb{R}^N} |w_n|^{2_s^*} dx \right)^{\frac{2}{2_s^*}}} \geq \tilde{S}_*. \quad (6.3)$$

We note that

$$\frac{p^2 + q^2}{(p^{\alpha} q^{\beta})^{\frac{2}{2_s^*}}} = \left(\frac{p}{q} \right)^{\frac{2\beta}{2_s^*}} + \left(\frac{p}{q} \right)^{-\frac{2\alpha}{2_s^*}}, \quad (6.4)$$

and we consider the function $g : \mathbb{R}_+ \rightarrow \mathbb{R}$ defined as

$$g(t) = t^{\frac{2\beta}{2_s^*}} + t^{-\frac{2\alpha}{2_s^*}}.$$

Then, it is easy to verify that g achieves its minimum at the point $t = \sqrt{\frac{\alpha}{\beta}}$, and

$$g\left(\sqrt{\frac{\alpha}{\beta}}\right) = \left(\frac{\alpha}{\beta}\right)^{\frac{\beta}{2_s^*}} + \left(\frac{\beta}{\alpha}\right)^{\frac{\alpha}{2_s^*}}. \quad (6.5)$$

Take p and q in (6.3) such that $\frac{p}{q} = \sqrt{\frac{\alpha}{\beta}}$, and we get

$$\left[\left(\frac{\alpha}{\beta}\right)^{\frac{\beta}{2_s^*}} + \left(\frac{\beta}{\alpha}\right)^{\frac{\alpha}{2_s^*}} \right] \frac{\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} w_n|^2 dx}{\left(\int_{\mathbb{R}^N} |w_n|^{2_s^*} dx \right)^{\frac{2}{2_s^*}}} \geq \tilde{S}_*$$

which gives

$$\left[\left(\frac{\alpha}{\beta}\right)^{\frac{\beta}{2_s^*}} + \left(\frac{\beta}{\alpha}\right)^{\frac{\alpha}{2_s^*}} \right] S_* \geq \tilde{S}_*. \quad (6.6)$$

Now, in order to conclude the proof, we consider a minimizing sequence $\{(u_n, v_n)\}$ for \tilde{S}_* . Let us define $z_n = p_n v_n$ where $p_n > 0$ is such that

$$\int_{\mathbb{R}^N} |u_n|^{2_s^*} dx = \int_{\mathbb{R}^N} |z_n|^{2_s^*} dx. \quad (6.7)$$

By using Young's inequality and (6.7), we can see that

$$\begin{aligned} \int_{\mathbb{R}^N} |u_n|^\alpha |z_n|^\beta dx &\leq \frac{\alpha}{2_s^*} \int_{\mathbb{R}^N} |u_n|^{\alpha+\beta} dx + \frac{\beta}{2_s^*} \int_{\mathbb{R}^N} |z_n|^{\alpha+\beta} dx \\ &= \int_{\mathbb{R}^N} |u_n|^{2_s^*} dx = \int_{\mathbb{R}^N} |z_n|^{2_s^*} dx. \end{aligned} \quad (6.8)$$

Therefore, by using (6.5), (6.8) and $\alpha + \beta = 2_s^*$, we can deduce that

$$\begin{aligned} \frac{\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u_n|^2 + |(-\Delta)^{\frac{s}{2}} v_n|^2 dx}{\left(\int_{\mathbb{R}^N} |u_n|^\alpha |v_n|^\beta dx \right)^{\frac{2}{2_s^*}}} &= \frac{p_n^{\frac{2\beta}{2_s^*}} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u_n|^2 + |(-\Delta)^{\frac{s}{2}} v_n|^2 dx}{\left(\int_{\mathbb{R}^N} |u_n|^\alpha |z_n|^\beta dx \right)^{\frac{2}{2_s^*}}} \\ &\geq p_n^{\frac{2\beta}{2_s^*}} \frac{\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u_n|^2 dx}{\left(\int_{\mathbb{R}^N} |u_n|^{2_s^*} dx \right)^{\frac{2}{2_s^*}}} + p_n^{\frac{2\beta}{2_s^*}} p_n^{-2} \frac{\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} z_n|^2 dx}{\left(\int_{\mathbb{R}^N} |z_n|^{2_s^*} dx \right)^{\frac{2}{2_s^*}}} \\ &\geq S_* \left(p_n^{\frac{2\beta}{2_s^*}} + p_n^{\frac{2\beta}{2_s^*} - 2} \right) = S_* g(p_n) \\ &\geq S_* g \left(\sqrt{\frac{\alpha}{\beta}} \right) = S_* \left[\left(\frac{\alpha}{\beta} \right)^{\frac{\beta}{2_s^*}} + \left(\frac{\beta}{\alpha} \right)^{\frac{\alpha}{2_s^*}} \right]. \end{aligned}$$

The thesis follows by passing to the limit in the above inequality. \square

In what follows, we prove the "critical version" of Lemma 3.1.

Lemma 6.2. *Let $\{(u_n, v_n)\} \subset \mathbb{H}_0$ be a Palais-Smale sequence for \mathcal{J}_0 at the level $d < \frac{s}{N} \tilde{S}_*^{\frac{N}{2s}}$. Then we have either*

- (i) $\|(u_n, v_n)\|_0 \rightarrow 0$, or
- (ii) *there exists a sequence $(y_n) \subset \mathbb{R}^N$ and $R, \gamma > 0$ such that*

$$\liminf_{n \rightarrow \infty} \int_{B_R(y_n)} (|u_n|^2 + |v_n|^2) dx \geq \gamma.$$

Proof. Assume that (ii) is not true. Then, for any $R > 0$, we get

$$\lim_{n \rightarrow \infty} \sup_{y \in \mathbb{R}^N} \int_{B_R(y)} |u_n|^2 dx = 0 = \lim_{n \rightarrow \infty} \sup_{y \in \mathbb{R}^N} \int_{B_R(y)} |v_n|^2 dx.$$

By using lemma 2.1, we can deduce that

$$u_n, v_n \rightarrow 0 \text{ in } L^r(\mathbb{R}^N) \quad \forall r \in (2, 2_s^*),$$

and in view of (1.6), we can see that $\int_{\mathbb{R}^N} Q(u_n, v_n) dx \rightarrow 0$.

Since $\{(u_n, v_n)\}$ is bounded, we have $\langle \mathcal{J}'_0(u_n, v_n), (u_n, v_n) \rangle \rightarrow 0$. Then, we have

$$\|(u_n, v_n)\|_0^2 - \int_{\mathbb{R}^N} (u_n^+)^{\alpha} (v_n^+)^{\beta} dx = o_n(1),$$

which implies that there exists $L \geq 0$ such that

$$\|(u_n, v_n)\|_0^2 \rightarrow L \text{ and } \int_{\mathbb{R}^N} (u_n^+)^{\alpha} (v_n^+)^{\beta} dx \rightarrow L. \quad (6.9)$$

Since $\mathcal{J}_0(u_n, v_n) \rightarrow d$, we can use (6.9) to deduce that $d = \frac{Ls}{N}$. From the definition of \tilde{S}_* we get

$$\|(u_n, v_n)\|_0^2 \geq \tilde{S}_* \left(\int_{\mathbb{R}^N} |u_n|^\alpha |v_n|^\beta dx \right)^{\frac{2}{2_s^*}} \geq \tilde{S}_* \left(\int_{\mathbb{R}^N} (u_n^+)^{\alpha} (v_n^+)^{\beta} dx \right)^{\frac{2}{2_s^*}},$$

which gives $L \geq \tilde{S}_* L^{\frac{2}{2s}}$. Now, if $L > 0$, we obtain $Nd = sL \geq s\tilde{S}_*^{\frac{N}{2s}}$ which provides a contradiction. Thus, $L = 0$ and (i) holds. \square

Now, we prove that the critical autonomous system admits a nontrivial solution.

Theorem 6.1. *The problem (6.1) has a weak solution.*

Proof. Since \mathcal{J}_0 has a mountain pass geometry, there exists $\{(u_n, v_n)\} \subset \mathbb{H}_0$ such that

$$\mathcal{J}_0(u_n, v_n) \rightarrow m_0 \text{ and } \mathcal{J}'_0(u_n, v_n) \rightarrow 0.$$

We aim to show that

$$m_0 < \frac{s}{N} \tilde{S}_*^{\frac{N}{2s}}. \quad (6.10)$$

Indeed, once proved (6.10), we can repeat the same arguments developed in the proof of Theorem 3.1 and to apply Lemma 6.2 instead of Lemma 3.1, to deduce the existence of a weak solution to (6.1). Therefore, by using the definition of m_0 , it is sufficient to prove that there exists $(u, v) \in \mathbb{H}_0$ such that

$$\max_{t \geq 0} \mathcal{J}_\varepsilon(tu, tv) < \frac{s}{N} \tilde{S}_*^{\frac{N}{2s}}.$$

Let $A, B > 0$ such that $\frac{A}{B} = \sqrt{\frac{\alpha}{\beta}}$. Then, in view of Lemma 6.1, we can deduce that

$$\tilde{S} = S \frac{(A^2 + B^2)}{(A^\alpha B^\beta)^{\frac{2}{2s}}}.$$

Fix $\eta \in C_0^\infty(\mathbb{R}^N)$ a cut-off function such that $0 \leq \eta \leq 1$, $\eta = 1$ on B_r and $\eta = 0$ on $\mathbb{R}^N \setminus B_{2r}$, where B_r denotes the ball in \mathbb{R}^N of center at origin and radius r .

For $\varepsilon > 0$, let us define $v_\varepsilon(x) = \eta(x)z_\varepsilon(x)$, where

$$z_\varepsilon(x) = \frac{\kappa \varepsilon^{\frac{N-2s}{2}}}{(\varepsilon^2 + |x|^2)^{\frac{N-2s}{2}}}$$

is a solution to

$$(-\Delta)^s u = S_* |u|^{2^*-2} u \text{ in } \mathbb{R}^N$$

and κ is a suitable positive constant depending only on N and s .

Now we set

$$u_\varepsilon = \frac{z_\varepsilon}{\left(\int_{\mathbb{R}^N} |z_\varepsilon|^{2^*} dx\right)^{\frac{1}{2^*}}}.$$

By performing similar calculations to those in [43] (see Proposition 21 and 22), we can see that

$$\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u_\varepsilon|^2 dx \leq S_* + O(\varepsilon^{N-2s}), \quad (6.11)$$

$$\int_{\mathbb{R}^N} |u_\varepsilon|^2 dx = \begin{cases} O(\varepsilon^{2s}) & \text{if } N > 4s \\ O(\varepsilon^{2s} |\log(\varepsilon)|) & \text{if } N = 4s \\ O(\varepsilon^{N-2s}) & \text{if } N < 4s \end{cases}, \quad (6.12)$$

and

$$\int_{\mathbb{R}^N} |u_\varepsilon|^q dx = \begin{cases} O(\varepsilon^{\frac{2N-(N-2s)q}{2}}) & \text{if } q > \frac{N}{N-2s} \\ O(|\log(\varepsilon)| \varepsilon^{\frac{N}{2}}) & \text{if } q = \frac{N}{N-2s} \\ O(\varepsilon^{\frac{(N-2s)q}{2}}) & \text{if } q < \frac{N}{N-2s}. \end{cases} \quad (6.13)$$

Thus, by using (Q6), we can note that

$$\mathcal{J}_0(tAu_\varepsilon, tBu_\varepsilon) \leq \left[\frac{t^2}{2}(A^2 + B^2)D_\varepsilon - \frac{t^{2_s^*}}{2_s^*}A^\alpha B^\beta \right] - \lambda t^{q_1} A^{q_1} B^{q_1} \int_{\mathbb{R}^N} |u_\varepsilon|^{q_1} dx =: h_\varepsilon(t),$$

where

$$D_\varepsilon = \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u_\varepsilon|^2 dx + \int_{\mathbb{R}^N} \max\{V_0, W_0\} u_\varepsilon^2 dx.$$

Let us denote by $t_\varepsilon > 0$ the maximum point of $h_\varepsilon(t)$. Since $h'_\varepsilon(t_\varepsilon) = 0$, we have

$$\bar{t}_\varepsilon = \left(\frac{D_\varepsilon(A^2 + B^2)}{(A^\alpha B^\beta)^{\frac{2}{2_s^*}}} \right)^{\frac{N-2s}{4s}} \geq t_\varepsilon > 0.$$

By using the fact that $h_\varepsilon(t)$ is increasing in $(0, \bar{t}_\varepsilon)$, we can deduce that

$$\mathcal{J}_0(tAu_\varepsilon, tBu_\varepsilon) \leq \frac{s}{N} \left(\frac{D_\varepsilon(A^2 + B^2)}{(A^\alpha B^\beta)^{\frac{2}{2_s^*}}} \right)^{\frac{N}{2s}} - \lambda t^{q_1} A^{q_1} B^{q_1} \int_{\mathbb{R}^N} |u_\varepsilon|^{q_1} dx.$$

Now, recalling that $(a+b)^r \leq a^r + r(a+b)^{r-1}b$ for any $a, b > 0$ and $r \geq 1$, we can see that

$$D_\varepsilon^{N/2s} \leq S_*^{N/2s} + O(\varepsilon^{N-2s}) + C_1 \int_{\mathbb{R}^N} |u_\varepsilon|^2 dx,$$

On the other hand, $h'_\varepsilon(t_\varepsilon) = 0$ and the mountain pass geometry of \mathcal{J}_ε , imply that there exists $\sigma > 0$ such that

$$t_\varepsilon \geq \sigma \text{ for any } \varepsilon > 0,$$

that is t_ε can be estimated from below by a constant independent of ε .

Then we have

$$\mathcal{J}_0(tAu_\varepsilon, tBu_\varepsilon) \leq \frac{s}{N} \tilde{S}_*^{\frac{N}{2s}} + O(\varepsilon^{N-2s}) + C_2 \int_{\mathbb{R}^N} |u_\varepsilon|^2 dx - \lambda C_3 \int_{\mathbb{R}^N} |u_\varepsilon|^{q_1} dx,$$

where $C_2, C_3 > 0$ are independent of ε and λ .

Now, we distinguish the following cases:

If $N > 4s$, then $q_1 > \frac{N}{N-2s}$. Hence, by using (6.12) and (6.13), we can see that

$$\sup_{t \geq 0} h_\varepsilon(t) \leq \frac{s}{N} \tilde{S}_*^{\frac{N}{2s}} + O(\varepsilon^{N-2s}) + O(\varepsilon^{2s}) - \lambda O(\varepsilon^{\frac{2N-(N-2s)q_1}{2}}).$$

Taking into account $\frac{2N-(N-2s)q_1}{2} < 2s < N-2s$, we get the thesis for ε small enough.

When $N = 4s$, then $q_1 \in (2, 4)$ and in particular $q_1 > \frac{N}{N-2s} = 2$, so from (6.12) and (6.13) we deduce that

$$\sup_{t \geq 0} h_\varepsilon(t) \leq \frac{s}{N} \tilde{S}_*^{\frac{N}{2s}} + O(\varepsilon^{2s}) + O(\varepsilon^{2s} |\log(\varepsilon)|) - \lambda O(\varepsilon^{4s-sq_1})$$

which implies (2.3) because of $\lim_{\varepsilon \rightarrow 0} \frac{\varepsilon^{4s-sq_1}}{\varepsilon^{2s(1+|\log(\varepsilon)|)}} = \infty$.

If $2s < N < 4s$ and $q_1 \in (\frac{4s}{N-2s}, 2_s^*)$, then $q_1 > \frac{N}{N-2s}$. Therefore we have

$$\sup_{t \geq 0} h_\varepsilon(t) \leq \frac{s}{N} \tilde{S}_*^{\frac{N}{2s}} + O(\varepsilon^{N-2s}) + O(\varepsilon^{N-2s}) - \lambda O(\varepsilon^{\frac{2N-(N-2s)q_1}{2}})$$

and we obtain the conclusion for ε sufficiently small since $\frac{2N-(N-2s)q_1}{2} < N - 2s$. If $2s < N < 4s$ and $q_1 \in (2, \frac{4s}{N-2s}]$, we argue as before and by using (6.13) we get

$$\sup_{t \geq 0} h_\varepsilon(t) \leq \begin{cases} \frac{s}{N} \tilde{S}_*^{\frac{N}{2s}} + O(\varepsilon^{N-2s}) - \lambda O(\varepsilon^{\frac{2N-(N-2s)q_1}{2}}) & \text{if } q_1 > \frac{N}{N-2s} \\ \frac{s}{N} \tilde{S}_*^{\frac{N}{2s}} + O(\varepsilon^{N-2s}) - \lambda O(|\log(\varepsilon)| \varepsilon^{\frac{N}{2}}) & \text{if } q_1 = \frac{N}{N-2s} \\ \frac{s}{N} \tilde{S}_*^{\frac{N}{2s}} + O(\varepsilon^{N-2s}) - \lambda O(\varepsilon^{\frac{(N-2s)q_1}{2}}) & \text{if } q_1 < \frac{N}{N-2s}. \end{cases}$$

Then, we can find $\lambda_0 > 0$ large enough such that for any $\lambda \geq \lambda_0$ and $\varepsilon > 0$ small it holds

$$\sup_{t \geq 0} h_\varepsilon(t) < \frac{s}{N} \tilde{S}_*^{\frac{N}{2s}}.$$

Taking into account the above estimates, we can infer that for any $\varepsilon > 0$ sufficiently small

$$\max_{t \geq 0} \mathcal{J}_0(tAu_\varepsilon, tBu_\varepsilon) \leq \max_{t \geq 0} h_\varepsilon(t) = h_\varepsilon(t_\varepsilon) < \frac{s}{N} \tilde{S}_*^{\frac{N}{2s}}.$$

□

Since we are interested in weak solutions of (1.8), we consider the re-scaled system

$$\begin{cases} (-\Delta)^s u + V(\varepsilon x)u = Q_u(u, v) + \frac{\alpha}{\alpha+\beta} |u|^{\alpha-2} u |v|^\beta & \text{in } \mathbb{R}^N \\ (-\Delta)^s u + W(\varepsilon x)v = Q_v(u, v) + \frac{\beta}{\alpha+\beta} |u|^\alpha |v|^{\beta-2} v & \text{in } \mathbb{R}^N \\ u, v > 0 & \text{in } \mathbb{R}^N. \end{cases} \quad (6.14)$$

Then, the corresponding functional $\mathcal{J}_\varepsilon : \mathbb{H}_\varepsilon \rightarrow \mathbb{R}$ is given by

$$\mathcal{J}_\varepsilon(u, v) = \frac{1}{2} \|(u, v)\|_\varepsilon^2 - \int_{\mathbb{R}^N} Q(u, v) dx - \frac{1}{\alpha + \beta} \int_{\mathbb{R}^N} (u^+)^{\alpha} (v^+)^{\beta} dx.$$

Clearly, the critical points of \mathcal{J}_ε belong to the Nehari manifold

$$\mathcal{M}_\varepsilon := \{(u, v) \in \mathbb{H}_\varepsilon \setminus \{(0, 0)\} : \langle \mathcal{J}'_\varepsilon(u, v), (u, v) \rangle = 0\}$$

and the ground state level is given by

$$m_\varepsilon := \inf_{(u, v) \in \mathcal{M}_\varepsilon} \mathcal{J}_\varepsilon(u, v) = \inf_{(u, v) \in \mathbb{H}_\varepsilon \setminus \{(0, 0)\}} \max_{t \geq 0} \mathcal{J}_\varepsilon(tu, tv) > 0.$$

As made in the previous sections, the Palais-Smale condition for the functional \mathcal{J}_ε is related with V_∞ and W_∞ . Then, as in Section 4, when $\max\{V_\infty, W_\infty\} < \infty$, we define the limit functional $\mathcal{J}_\infty : \mathbb{H}_0 \rightarrow \mathbb{R}$ by setting

$$\begin{aligned} \mathcal{J}_\infty(u, v) := & \frac{1}{2} \left(\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 + |(-\Delta)^{\frac{s}{2}} v|^2 dx + \int_{\mathbb{R}^N} (V_\infty u^2 + W_\infty v^2) dx \right) \\ & - \int_{\mathbb{R}^N} Q(u, v) dx - \frac{1}{\alpha + \beta} \int_{\mathbb{R}^N} (u^+)^{\alpha} (v^+)^{\beta} dx, \end{aligned}$$

and its ground state level

$$m_\infty := \inf_{(u, v) \in \mathbb{H}_0 \setminus \{(0, 0)\}} \max_{t \geq 0} \mathcal{J}_\infty(tu, tv) > 0.$$

If $\max\{V_\infty, W_\infty\} = \infty$, we set $m_\infty := \infty$.

Since the map $(u, v) \mapsto \int_{\mathbb{R}^N} (u^+)^{\alpha} (v^+)^{\beta} dx$ is positively 2_s^* -homogeneous, the arguments developed in Section 4 permit to deduce a compactness result for the functional \mathcal{J}_ε . More precisely, following the lines of the proofs of Theorem 4.1 and Corollary 4.1, by using Lemma 6.2 instead of Lemma 3.1, we can prove that the following result holds.

Theorem 6.2. *The functional \mathcal{J}_ε constrained to \mathcal{M}_ε satisfies the $(PS)_d$ -condition at any level $d < \min\{m_\infty, \frac{s}{N} \tilde{S}_*^{\frac{N}{2s}}\}$. Moreover, critical points of \mathcal{J}_ε constrained to \mathcal{M}_ε are critical points of \mathcal{J}_ε in \mathbb{H}_ε .*

We conclude this section giving our second multiplicity result. Since many calculations made in Section 5 can be adapted in this context, we present only a sketch of the proof.

Proof of Theorem 1.2. We proceed as in the proof of Theorem 1.1. Fix $\delta > 0$ and choose $\eta \in C_0^\infty(\mathbb{R}, [0, 1])$ such that $\eta(t) = 1$ if $0 \leq t \leq \frac{\delta}{2}$ and $\eta(t) = 0$ if $t \geq \delta$. Let $(\tilde{w}_1, \tilde{w}_2) \in \mathbb{H}_0$ be the solution of (6.1) given by Theorem 6.1. For any $y \in M$, we define

$$\tilde{\Psi}_{i,\varepsilon,y}(x) := \eta(|\varepsilon x - y|) \tilde{w}_i \left(\frac{\varepsilon x - y}{\varepsilon} \right), \quad i = 1, 2,$$

and we introduce the map $\tilde{\Phi}_\varepsilon(y) := (\tilde{t}_\varepsilon \tilde{\Psi}_{1,\varepsilon,y}, \tilde{t}_\varepsilon \tilde{\Psi}_{2,\varepsilon,y})$, where \tilde{t}_ε is the unique positive number satisfying

$$\max_{t \geq 0} \mathcal{J}_\varepsilon(t \tilde{\Psi}_{1,\varepsilon,y}, t \tilde{\Psi}_{2,\varepsilon,y}) = \mathcal{J}_\varepsilon(\tilde{t}_\varepsilon \tilde{\Psi}_{1,\varepsilon,y}, \tilde{t}_\varepsilon \tilde{\Psi}_{2,\varepsilon,y}).$$

As in Section 5, we can see that

$$\lim_{\varepsilon \rightarrow 0^+} \mathcal{J}_\varepsilon(\tilde{\Phi}_\varepsilon(y)) = m_0 \quad \text{uniformly for } y \in M.$$

Moreover, denoted by $\Upsilon : \mathbb{R}^N \rightarrow \mathbb{R}^N$ the function defined in Section 4, we can define the barycenter map $\tilde{\beta}_\varepsilon : \mathcal{M}_\varepsilon \rightarrow \mathbb{R}^N$ given by

$$\tilde{\beta}_\varepsilon(u, v) := \frac{\int_{\mathbb{R}^N} \Upsilon(\varepsilon x) (|u(x)|^2 + |v(x)|^2) dx}{\int_{\mathbb{R}^N} (|u(x)|^2 + |v(x)|^2) dx}.$$

Then, it is easy to prove that

$$\lim_{\varepsilon \rightarrow 0^+} \tilde{\beta}_\varepsilon(\Phi_\varepsilon(y)) = y \quad \text{uniformly for } y \in M$$

and

$$\lim_{\varepsilon \rightarrow 0^+} \sup_{(u,v) \in \tilde{\Sigma}_\varepsilon} \text{dist}(\tilde{\beta}_\varepsilon(u, v), M_\delta) = 0,$$

where

$$\tilde{\mathcal{M}}_\varepsilon := \{(u, v) \in \mathcal{M}_\varepsilon : \mathcal{J}_\varepsilon(u, v) \leq m_0 + \tilde{h}(\varepsilon)\}$$

and $\tilde{h} : [0, \infty) \rightarrow [0, \infty)$ satisfies $\tilde{h}(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0^+$.

As a consequence, there exists $\varepsilon_\delta > 0$ such that, for any $\varepsilon \in (0, \varepsilon_\delta)$, the diagram

$$M \xrightarrow{\tilde{\Phi}_\varepsilon} \tilde{\mathcal{M}}_\varepsilon \xrightarrow{\tilde{\beta}_\varepsilon} M_\delta$$

is well defined and $\tilde{\beta}_\varepsilon \circ \tilde{\Phi}_\varepsilon$ is homotopically equivalent to the embedding $\iota : M \rightarrow M_\delta$. Therefore, $\text{cat}_{\tilde{\mathcal{M}}_\varepsilon}(\tilde{\mathcal{M}}_\varepsilon) \geq \text{cat}_{M_\delta}(M)$. By using Theorem 6.2 and $m_0 < \frac{s}{N} \tilde{S}_{*}^{\frac{N}{2s}}$, we may suppose that ε_δ is so small such that \mathcal{J}_ε satisfies the Palais-Smale condition in $\tilde{\mathcal{M}}_\varepsilon$. Then the proof goes as in the subcritical case by using Ljusternik-Schnirelmann theory. \square

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